

# Uncertainty

CS 580

Intro to Artificial Intelligence

# Sensor uncertainty

We've been assuming we can tell what state we're currently in with perfect accuracy. This is often not true!

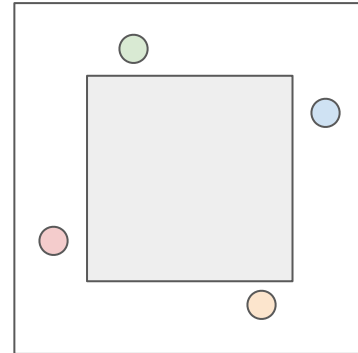
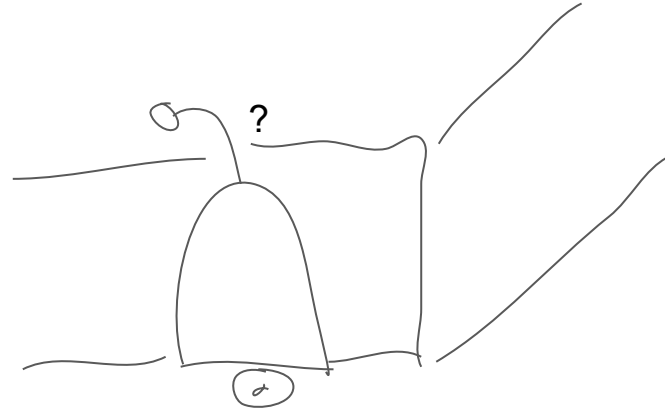
- Is it going to rain in the next hour?
- Do I have allergies or a cold?
- How much time do I have before my battery runs out?

Probabilities to the rescue!

$S = \{\text{'loc': 'A', 'A-clean': True, 'B-clean': False}\}$  vs

$S = \{\text{'loc-is-A': 0.5, 'A-clean': 0.9, 'B-clean': 0.2}\}$

We can ask questions like: “What’s the probability that both A and B are clean?”



# Probabilistic Inference

Represent components of state as **Random Variables**:

$$p(X_i = \text{TRUE}) = p, \quad 0 \leq p \leq 1$$

Random Variables can be **discrete** or continuous

Raining  $\in \{\text{TRUE}, \text{FALSE}\}$

Battery  $\in [0, \infty)$

The function that determines the **probability** value is called the “distribution” of that RV

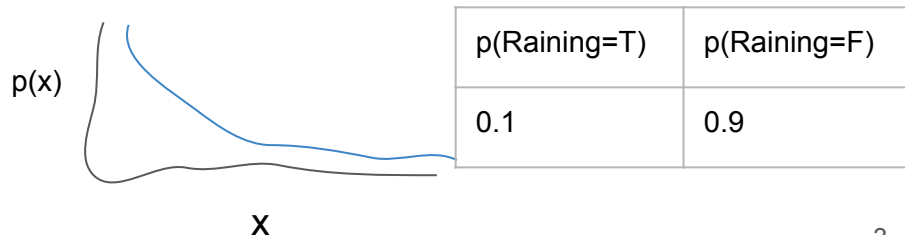
How do we know the distribution? Two ways:

**Sampling** by observing many times

- Expected utility in RL

Provided as part of the problem description

- $p(\text{Raining}) = \langle 0.1, 0.9 \rangle$
- $p(\text{Battery} > x) = e^{-x}$



# Rules of Probability

1. The probability that a RV takes on some value is always between 0 and 1

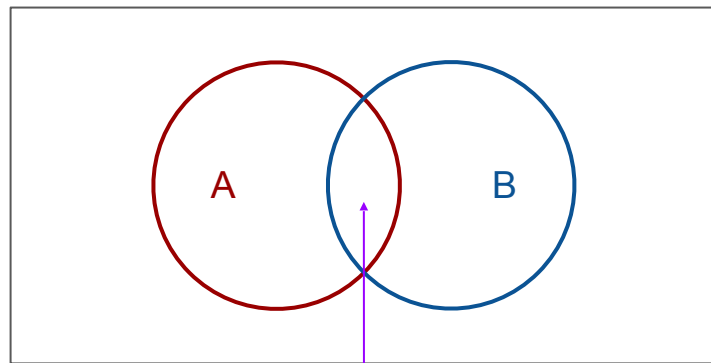
$$0 \leq p(X = x) \leq 1$$

2. Probability of deterministic events

$$p(\text{TRUE}) = 1, \quad p(\text{FALSE}) = 0$$

3. Additivity

$$p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$$



A and B

$$p(A = \text{TRUE} \vee A = \text{FALSE}) = p(A = \text{TRUE}) + p(A = \text{FALSE}) - p(A = \text{TRUE} \wedge A = \text{FALSE})$$

$$1 = p(A = \text{TRUE}) + p(A = \text{FALSE}) - 0$$

$$1 - p(A = \text{FALSE}) = p(A = \text{TRUE})$$

# Interactions between Random Variables

With multiple interacting RVs, the probability distribution that includes **all** of them together is called the **joint distribution**

**Example:**  $p(\text{Cavity}, \text{Toothache}, \text{Catch})$

## Notation

$p(X)$ : distribution, function or table

$p(X=x)$ : probability, single number

	Toothache		$\neg$ Toothache	
	Catch	$\neg$ Catch	Catch	$\neg$ Catch
Cavity	0.108	0.012	0.072	0.008
$\neg$ Cavity	0.016	0.064	0.144	0.576

**Facts:** table sums to 1, all combinations of all RVs, one cell per “configuration”

$$p(\text{Toothache}=F, \text{Cavity}=F, \text{Catch}=F) = 0.576$$

# Events & Marginalization

- An **event** is a setting of some subset of random variables
- You can use the joint distribution to compute the probability of any event by **adding up** all the table entries that correspond with the given configuration

## Marginalization

$$p(\text{Cav}=T) = 0.108 + 0.012 + 0.072 + 0.008 \\ = 0.2$$

	Tooth		$\neg$ Tooth	
	Cat	$\neg$ Cat	Cat	$\neg$ Cat
Cav	0.108	0.012	0.072	0.008
$\neg$ Cav	0.016	0.064	0.144	0.576

$$p(\text{Cav}=T \text{ or } \text{Tooth}=T) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$$

$$p(\text{Cav}=T \text{ and } \text{Tooth}=T) = 0.108 + 0.012 = 0.12$$

# Conditional Probability (1)

- When one Random Variable **impacts** the value of another  
“If  $X=v$ , what is the probability of  $Y=w$ ?”

$$p(Y = w \mid X = v) = \frac{p(Y = w \text{ and } X = v)}{p(X = v)}$$

“Conditioned variable”

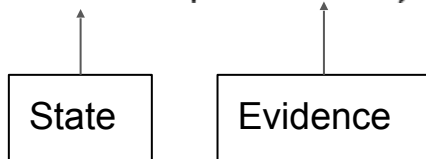
## Notation

$p(Y|X=v)$ : distribution, function or table  
 $p(Y=w|X=v)$ : probability, single number

# Conditional Probability (2)

Typically, an agent wants to know conditional probabilities

$$p(S = s \mid E = e)$$



	Tooth		$\neg$ Tooth	
	Cat	$\neg$ Cat	Cat	$\neg$ Cat
Cav	0.108	0.012	0.072	0.008
$\neg$ Cav	0.016	0.064	0.144	0.576

Example:

$$p(\text{Cav}=\text{T} \mid \text{Tooth}=\text{T}) = p(\text{Cav}=\text{T} \text{ and } \text{Tooth}=\text{T})/p(\text{Tooth}=\text{T})$$

$$= (0.108+0.012)/(0.108+0.012+0.016+0.064) = 0.6$$

$$p(\text{Cav}=\text{F} \mid \text{Tooth}=\text{T}) = (0.016+0.064)/(0.108+0.012+0.016+0.064) = 0.4$$



# Normalization

$$\begin{aligned} p(\text{Cav}=\text{T} \mid \text{Tooth}=\text{T}) &= 0.6 \\ p(\text{Cav}=\text{F} \mid \text{Tooth}=\text{T}) &= 0.4 \end{aligned}$$

Notice that the two probabilities summed to 1 across Cavity (**not** Toothache)

In general, we don't have to know/compute the denominator, we can “normalize”

$$p(X \mid E = e) = \frac{p(X, E = e)}{p(E = e)} = \alpha \cdot p(X, E = e)$$

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$$1 = \alpha \cdot p(X = \text{TRUE}, E = e) + \alpha \cdot p(X = \text{FALSE}, E = e)$$

$$1 = \alpha [p(X = \text{TRUE}, E = e) + p(X = \text{FALSE}, E = e)]$$

$$\alpha = \frac{1}{[p(X = \text{TRUE}, E = e) + p(X = \text{FALSE}, E = e)]}$$

# Hidden Variables

Frequently our agent won't have settings for **all** of the random variables.

Solution? Sum them out!

$$p(X \mid E = e) = \alpha \cdot p(X, E = e) = \alpha \sum_{h \in H} p(X, E = e, H = h)$$

In the previous example, **Catch** was a **hidden** variable:

$$p(\text{Cav}=\text{T} \mid \text{Tooth}=\text{T}) = \alpha p(\text{Cav}=\text{T}, \text{Tooth}=\text{T}, \text{Cat}=\text{T}) + \alpha p(\text{Cav}=\text{T}, \text{Tooth}=\text{T}, \text{Cat}=\text{F})$$

# Example - marginalization and normalization

What's the probability you have a cavity if you don't have a toothache?

$$\begin{aligned} p(\text{Cav}|\text{Tooth}=F) &= \alpha p(\text{Cav}, \text{Tooth}=F) = \alpha \sum_{h=\{T,F\}} p(\text{Cav}, \text{Tooth}=F, \text{Catch}=h) \\ &= \alpha [p(\text{Cav}, \text{Tooth}=F, \text{Cat}=T) + p(\text{Cav}, \text{Tooth}=F, \text{Cat}=F)] \\ &= \alpha [ <0.072, 0.144> + <0.008, 0.576>] \\ &= \alpha <0.08, 0.72> \end{aligned}$$

Must sum to 1, so

$$\alpha * (0.08 + 0.72) = \alpha * (0.8) = 1$$

$$\alpha = 1.25$$

$$p(\text{Cav}|\text{Tooth}=F) = <0.1, 0.9>$$

		Tooth		¬Tooth	
		Cat	¬Cat	Cat	¬Cat
Cav	¬Cav	0.108	0.012	0.072	0.008
		0.016	0.064	0.144	0.576

# Conditioning

- Given full joint probability, agent can compute anything about the RVs!
- Usually, agent only has access to some conditional probabilities, not their joint distribution
- Can get around this by marginalizing and leveraging the definition of conditional probability

Ex: How can we get  $p(X)$  (marginal) if we don't know  $p(X,Y)$ , just  $p(X|Y)$  and  $p(Y)$ ?

Cond. Prob. def.  $\longrightarrow p(X | Y) = \frac{p(X, Y)}{p(Y)}$

$$p(X | Y) \cdot p(Y) = p(X, Y)$$

Marginalize Y  $\longrightarrow \sum_y p(X | Y = y)p(Y = y) = \sum_y p(X, Y = y) = p(X)$

# Independence

Two random variables are **independent** if and only if their joint probability is **the same** as the product of their marginals

$$\begin{aligned}A \perp B &\iff \\p(A, B) &= p(A)p(B) \\p(A \mid B) &= p(A)\end{aligned}$$

Similarly, two random variables can be **conditionally independent** given a third

$$\begin{aligned}A \perp B \mid C &\iff \\p(A, B \mid C) &= p(A \mid C)p(B \mid C) \\p(A \mid B, C) &= p(A \mid C)\end{aligned}$$

# Independence Example

Are Catch and Toothache independent?

$$p(\text{Catch}=T, \text{Toothache}=T) = 0.124$$

$$\begin{aligned} p(\text{Cat}=T) * p(\text{Tooth}=T) &= (0.108 + 0.016 + 0.072 + 0.144) * (0.108 + 0.012 + 0.016 + 0.064) \\ &= (0.34) * (0.2) = 0.068 \neq 0.124 \text{ (NOT INDEPENDENT)} \end{aligned}$$

What about conditioned on Cavity?

$$p(\text{Cat}=T, \text{Tooth}=T | \text{Cav}=T) = 0.54$$

$$p(\text{Cat}=T | \text{Cav}=T) * p(\text{Tooth}=T | \text{Cav}=T) = 0.9 * 0.6 = 0.54$$

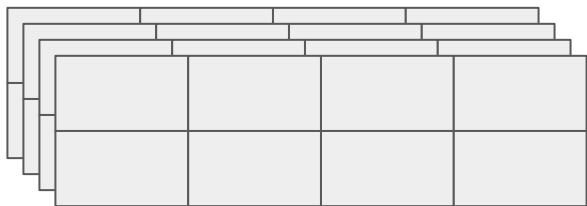
... also checks out for each setting of Cat, Cav, and Tooth  
**(CONDITIONALLY INDEPENDENT given Cavity)**

	Tooth		$\neg$ Tooth	
	Cat	$\neg$ Cat	Cat	$\neg$ Cat
Cav	0.108	0.012	0.072	0.008
$\neg$ Cav	0.016	0.064	0.144	0.576

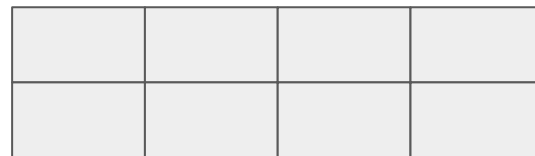
# Factoring the joint probability with independence

- Suppose we add Weather as a RV to the Cav,Cat,Tooth model, where whether can take on 4 values: {Sunny, Rainy, Foggy, Snowy}

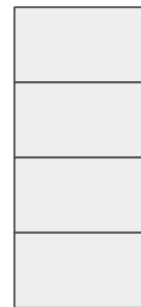
$$p(\text{Cav,Tooth,Catch,Weather}) = p(\text{Cav,Tooth,Catch}) * p(\text{Weather})$$



$2 \times 2 \times 2 \times 4 = 32$  cells



$2 \times 2 \times 2 + 4 = 12$  cells



# Bayes Rule

Additional TRUE FACT about conditional probabilities

$$p(A \mid B) = \frac{p(B \mid A) \cdot p(A)}{p(B)}$$

Follows by plugging in definition of conditional probability

$$p(B \mid A) \cdot p(A) = p(A, B) = p(A \mid B)p(B)$$

This generalizes to more than two RVs, and gives us the **product rule**

$$p(A, B, C) = p(A, B \mid C) \cdot p(C) = p(A \mid B, C) \cdot p(B \mid C) \cdot p(C)$$

$$p(X_1, X_2, \dots, X_d) = p(X_1 \mid X_2, \dots, X_d) \cdot p(X_2 \mid X_3, \dots, X_d) \cdot \dots \cdot p(X_{d-1} \mid X_d) \cdot p(X_d)$$

**Note:** this is true for **any** ordering of the  $X_j$ !



# Using Bayes Rule

We can use Bayes Rule to “flip” the conditioning variable

$$p(\text{CAUSE} \mid \text{EFFECT}) = p(\text{EFFECT} \mid \text{CAUSE}) \frac{p(\text{CAUSE})}{p(\text{EFFECT})}$$

Difficult to measure

Easy to measure

No need to measure  
(normalize!)

Easy to measure

Example: Medical Diagnosis “What’s the probability I have the flu given I have a cough?”

$$p(\text{Flu} \mid \text{Cough}=\text{T}) = \alpha \langle p(\text{Cough}=\text{T} \mid \text{Flu}=\text{T}) p(\text{Flu}=\text{T}), p(\text{Cough}=\text{T} \mid \text{Flu}=\text{F}) p(\text{Flu}=\text{F}) \rangle$$

# Incorporating multiple pieces of evidence

“What’s the probability of Cavity given Catch and Toothache?”

Applying bayes rule and normalizing:

$$p(\text{Cav} \mid \text{Cat}, \text{Tooth}) = \alpha p(\text{Tooth}, \text{Cat} \mid \text{Cav}) p(\text{Cav})$$

But since Catch and Tooth are conditionally independent given Cavity

$$p(\text{Cav} \mid \text{Cat}, \text{Tooth}) = \alpha p(\text{Tooth} \mid \text{Cav}) p(\text{Cat} \mid \text{Cav}) p(\text{Cav})$$

**IF** our **evidence** variables are conditionally independent from one another **given** the **cause** variable, we can significantly simplify things!

# Summary and preview

## Wrapping up

- To handle uncertain sensors, state now describes **probabilities**
- The joint probability distribution contains **all** the information we need to answer any question about any subset of the random variables it describes
- Marginalization, Normalization, and Bayes Rule are all “probability tricks” we can use to simplify/compute specific probabilities

## Next time

- Detailed example: Avoid the wumpus!